FULL LARGE DEVIATION PRINCIPLE FOR BENEDICKS-CARLESON QUADRATIC MAPS: ACIP AS A REFERENCE

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ABSTRACT. Since the pioneering works of Jakobson and Benedicks & Carleson, it has been known that positive measure set of quadratic maps admit invariant probability measures absolutely continuous with respect to Lebesgue. These measures allow one to statistically predict the asymptotic fate of Lebesgue almost every initial condition. Estimating "Séjour probabilities" of empirical distributions before they settle to equilibrium requires a fairly good control over large parts of the phase space. We use the sub-exponential slow recurrence condition of Benedicks & Carleson, to build various induced Markov maps and associated towers, to which the absolutely continuous measures can be lifted. Considering these various lifts altogether enables us to obtain a control of recurrence, sufficient to establish a level 2 large deviation principle, with the absolutely continuous measures as references. This full result encompasses dynamics far from equilibrium, and thus significantly extends presently known local large deviations results for quadratic maps.

1. INTRODUCTION

Let $X = [-1, 1]$, and let $f_a \colon X \circlearrowleft$ be the quadratic map given by $f_a x = 1 - ax^2$, where $0 < a \leq 2$. The abundance of parameters in this family for which "chaotic dynamics" occur has been known since the pioneering works of Jakobson [19] and Benedicks & Carleson [5, 6]: there exists a set of a-values near 2 with positive Lebesgue measure for which the corresponding $f = f_a$ admits an invariant probability measure μ that is absolutely continuous with respect to Lebesgue (acip). By a classical theorem, for Lebesgue a.e. x the empirical distribution $\delta_x^n = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i x}$ converges to μ. The theory of large deviations aims to provide exponential bounds on the probability that δ_x^n stay away from μ.

Large deviations questions have been addressed for various stochastic processes (See e.g. [17, 18]). For dynamical systems, one cannot expect a full large deviation principle without strong assumptions [15, 21, 27, 33]. For the quadratic map, the enemy is the critical point $x = 0$. Up to now, only local large deviations results are known [20, 23, 28], and a full result which encompasses dynamics far from equilibrium is still missing. Our aim here is to write down a simple set of conditions satisfied on a positive measure set in parameter space, and to show that when these conditions are met, then a full large deviation principle holds.

We formulate our conditions as follows: Let $\lambda = \frac{9}{10} \log 2$ and $\alpha = \frac{1}{100}$.

(A1) $f = f_a$ where a is sufficiently near 2;
(A2) $|(f^n)'(f0)| \geq e^{\lambda n} \; \forall n \geq 0$;
(A3) $|f^n 0| \geq e^{-\alpha \sqrt{n}} \; \forall n \geq 1$;

(A4) f is topologically mixing on $[f^2 0, f 0]$.

Near 2, the abundance of parameters for which (A2) is satisfied was proved by Benedicks & Carleson [6]. For their parameters, (A4) holds (See [34] Lemma 2.1). The abundance of parameters for which (A3) is satisfied was proved by Benedicks & Young [7], and earlier by Benedicks & Carleson [5] under slightly different hypotheses. The parameter sets they constructed have 2 as a full Lebesgue density point. Hence, given $a_0 < 2$ arbitrarily near 2 there is a set $A \subset [a_0, 2]$ with positive Lebesgue measure such that for all $a \in A$, f_a satisfies (A2)-(A4).

In what follows we assume $f = f_a$ satisfies (A1)-(A4). Then f admits an acip μ. Let $\mathcal{M}$ denote the space of Borel probability measures on X endowed with the topology of weak convergence. We say the *large deviation principle holds for* (f, μ) if there exists a lower semi-continuous function called a *rate function* $I \colon \mathcal{M} \to [0, \infty]$ such that:

$$\varliminf_{n \to \infty} \frac{1}{n} \log \mu\{x \in X \colon \delta_x^n \in \mathcal{G}\} \geq -\inf\{I(\nu) \colon \nu \in \mathcal{G}\}$$

for any open set $\mathcal{G} \subset \mathcal{M}$, and

$$\varlimsup_{n \to \infty} \frac{1}{n} \log \mu\{x \in X \colon \delta_x^n \in \mathcal{K}\} \leq -\inf\{I(\nu) \colon \nu \in \mathcal{K}\}$$

for any closed set $\mathcal{K} \subset \mathcal{M}$. Here, the Dirac measure at x is written δ_x and $\delta_x^n = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i x}$. If the large deviation principle holds, then the rate function is unique.

Let $\mathcal{M}_f$ denote the set of f-invariant Borel probability measures. Define a mean Lyapunov exponent of $\nu \in \mathcal{M}_f$ by $\lambda(\nu) = \int \log |f'| d\nu$. This is well-defined, and $\lambda(\nu) > 0$ holds for any $\nu \in \mathcal{M}_f$ [11, 26]. Let $h(\nu)$ denote the entropy of ν, and define a free energy function $F \colon \mathcal{M} \to \mathbb{R}$ by

$$F(\nu) = \begin{cases} h(\nu) - \lambda(\nu) & \text{if } \nu \in \mathcal{M}_f; \\ -\infty & \text{otherwise.} \end{cases}$$

By Ruelle's inequality [29], $F(\nu) \leq 0$ and the equality holds only if $\nu = \mu$ [22].

Theorem. *If $f = f_a$ satisfies* (A1)-(A4), *then the large deviation principle holds for (f, μ). The rate function I is the lower semi-continuous regularization[1] of $-F$, i.e.,*

$$I(\nu) = -\inf_{\mathcal{G}} \sup\{F(\eta) \colon \eta \in \mathcal{G}\},$$

where the infimum is taken over all neighborhoods $\mathcal{G} \subset \mathcal{M}$ of ν.

We state a corollary which follows from a general theory on large deviations (See e.g. [16]), and use it to compare our result with the previous related ones. Let $C(X)$ denote the space of all continuous functions on X. For $\varphi \in C(X)$, let $S_n \varphi = \sum_{i=0}^{n-1} \varphi \circ f^i$. Let

$$c_\varphi := \inf_{x \in X} \varliminf_{n \to \infty} \frac{1}{n} S_n \varphi(x) \quad \text{and} \quad d_\varphi := \sup_{x \in X} \varlimsup_{n \to \infty} \frac{1}{n} S_n \varphi(x).$$

The compactness of $\mathcal{M}_f$ implies $c_\varphi = \min\{\nu(\varphi) \colon \nu \in \mathcal{M}_f\}$ and $d_\varphi = \max\{\nu(\varphi) \colon \nu \in \mathcal{M}_f\}$, where $\nu(\varphi) = \int \varphi d\nu$. Let $\log 0 = -\infty$ and $\sup \emptyset = -\infty$. For $t \in \mathbb{R}$, define

$$F_\varphi(t) = \sup\{F(\nu) \colon \nu \in \mathcal{M}_f, \nu(\varphi) = t\}.$$

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[1]We remark that the lower semi-continuous regularization is not removable to get the rate function. In fact, the free energy may not be upper semi-continuous [11].

Without loss of generality we may assume $c_\varphi < d_\varphi$ (Otherwise, $F_\varphi = -\infty$ except at one point). Then F_φ is bounded and concave on the interval (c_φ, d_φ).

Corollary 1. (Contraction principle) *Let $a, b \in \mathbb{R}$ be such that $a < b$. If $a \neq d_\varphi$ or $b \neq c_\varphi$, then*

$$\lim_{n \to \infty} \frac{1}{n} \log \mu \left\{ a \leq \frac{1}{n} S_n \varphi \leq b \right\} = \max_{a \leq t \leq b} F_\varphi(t).$$

Our theorem is the first full large deviations result for a positive measure set of quadratic maps, despite a large number of papers over the past thirty years dedicated to stochastic properties of chaotic dynamics in one-dimensional maps. Up to now, only local results are known, which claim the existence of the above limit in the case where φ is Hölder continuous and a, b are near the mean $\mu(\varphi)$ [20, 23, 28].

The next corollary follows from Varadhan's integral lemma [[16], pp.137] and the convex duality of Fenchel-Legendre transforms [[16], pp.152]. See also [30, 31].

Corollary 2. (Variational principle of Gibbs type) *For any $\varphi \in C(X)$, the limit*

$$P(\varphi) := \lim_{n \to \infty} \frac{1}{n} \log \mu(e^{S_n \varphi})$$

exists. In addition,

$$P(\varphi) = \max_{\nu \in \mathcal{M}} \{ \nu(\varphi) - I(\nu) \} \quad \text{for all } \varphi \in C(X),$$

and

$$I(\nu) = \max_{\varphi \in C(X)} \{ \nu(\varphi) - P(\varphi) \} \quad \text{for all } \nu \in \mathcal{M}.$$

For a broad class of nonuniformly hyperbolic systems including the quadratic maps we treat here, tower methods have been used to draw their statistical properties in a unified way [35, 36]. A proof of the theorem also relies on the construction of induced Markov maps and associated towers. However, there are some important differences in how towers are used.

For our quadratic maps, towers have already been constructed (see e.g. [3, 12, 34, 35]), for which the decay rate of the tail of return times is exponential. We stress that exponential tails do not necessarily imply the full large deviation principle, primarily because probabilities of rare events carried on the tails are unaccounted for. For instance, for certain Markov processes it is well-known [4, 17, 18] that exponential tails of return times are in general not sufficient to ensure a full large deviation principle. They only imply a local large deviation result [18], which is similar to the result in [28]. A full large deviation principle for stationary processes have been established under very strong mixing conditions [9, 10], which cannot be expected for dynamical systems.

In [13], sufficient conditions on the "shape" of towers were introduced to ensure a full large deviation principle (with Lebesgue as a reference). Nevertheless, for our quadratic maps it is hard to construct such "ideal towers", apart from very special cases (e.g. Misiurewicz maps). So, our idea is to abandon working with a single tower, and instead construct various induced Markov maps and associated towers. We use them altogether to obtain an upper exponential bound, on the probability that time averages of continuous functions stay away from their spatial averages. Comparing this upper bound with a lower exponential one, which is a direct consequence of [13, 25, 34], we establish the large deviation principle. We remark that upper and lower large deviation bounds were obtained in [33] in a very general setting, and in [1]

for nonuniformly expanding maps. These bounds are not comparable and so not sufficient to conclude the large deviation principle.

A proof of the theorem is briefly outlined as follows. Given $d \geq 1$, functions $\varphi_1, \ldots, \varphi_d$ on X and $b_1, \ldots, b_d \in \mathbb{R}$, define

$$\overline{R}(\varphi_1, \ldots, \varphi_d; b_1, \ldots, b_d) = \overline{\lim_{n \to \infty}} \frac{1}{n} \log \mu \left\{ \frac{1}{n} S_n \varphi_j \geq b_j, \ \ j = 1, \ldots, d \right\},$$

and

$$\underline{R}(\varphi_1, \ldots, \varphi_d; b_1, \ldots, b_d) = \varliminf_{n \to \infty} \frac{1}{n} \log \mu \left\{ \frac{1}{n} S_n \varphi_j > b_j, \ \ j = 1, \ldots, d \right\}.$$

All our effort is dedicated to proving the next

Proposition. *Let $f = f_a$ satisfy (A1)-(A4). Let $d \geq 1$ and let $\varphi_1, \ldots, \varphi_d$ be a collection of Lipschitz continuous functions on X, and let $b_1, \ldots, b_d \in \mathbb{R}$. For any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then there exists an ergodic measure $\sigma \in \mathcal{M}_f$ such that:*

(1)
$$\frac{1}{n} \log \mu \left\{ \frac{1}{n} S_n \varphi_j \geq b_j, \ j = 1, \ldots, d \right\} \leq (1 - 200\varepsilon) F(\sigma) + \varepsilon;$$

(2)
$$\sigma(\varphi_j) \geq b_j - \sqrt{\varepsilon}, \quad j = 1, \ldots, d.$$

It then follows that

(3) $\overline{R}(\varphi_1, \ldots, \varphi_d; b_1, \ldots, b_d) \leq \lim_{\varepsilon \to +0} \sup \left\{ F(\nu) \colon \nu \in \mathcal{M}_f, \ \nu(\varphi_j) \geq b_j - \sqrt{\varepsilon}, \ \ j = 1, \ldots, d \right\}.$

Meanwhile, by a result of [25, 34], the density of μ is uniformly bounded away from zero on $[f^2 0, f 0]$. Hence, the lower bound obtained in [13] with Lebesgue as a reference translates into a lower bound with μ as a reference, namely

(4) $\underline{R}(\varphi_1, \ldots, \varphi_d; b_1, \ldots, b_d) \geq \sup \left\{ F(\nu) \colon \nu \in \mathcal{M}, \ \nu(\varphi_j) > b_j, \ \ j = 1, \ldots, d \right\}.$

Now, observe that the weak topology on $\mathcal{M}$ has a countable base which is generated by open sets of the form $\{\nu \in \mathcal{M} \colon \nu(\varphi_j) > b_j, \ j = 1, \ldots, d\}$, where $d \geq 1$, each φ_j is Lipschitz continuous and $b_j \in \mathbb{R}$. Hence, (3) (4) imply the theorem.

Our strategy for the proof of the proposition is to construct an induced map and an associated tower *parametrized by ε*, and use them to find a horseshoe which carries an ergodic measure with the properties as in the statement. At this point, we make an important use of the sub-exponential slow recurrence condition (A3).

The rest of this paper consists of two sections. In Sect.2 we develop preliminary estimates. We modify the classical binding argument and the return time estimate [5, 6], in such a way that allows us to treat an arbitrarily small ε. In Sect.3 we prove the proposition. A crucial estimate is Lemma 3.8, which roughly says that *any partition element of the tower is approximated by points which quickly return to the base of the tower*. To equip our tower with this property, we construct an induced map on a Cantor set, which consists of points slow recurrent to the critical point. This construction is inspired by that of Benedicks & Young for Hénon-like attractors [8].

To keep the brevity of this paper we stay away from generalizations. Our arguments and results can be generalized with no essential difficulty to C^2 Collet-Eckmann unimodal maps with non-flat critical point, for which the recurrence of the critical orbit is sub-exponential. It is known [2] that almost every stochastic quadratic map satisfies these two conditions. Lastly,

with some additional arguments it is possible for our quadratic maps to establish the large deviation principle with Lebesgue measure as a reference [14].

2. Preliminary estimates

In this section we develop preliminary estimates needed for the proof of the proposition. We develop a binding argument for recovering expansion and prove a return time estimate. Original ideas for these can be found in [5, 6]. We modify them to treat an arbitrarily small ε.

2.1. **Bounded distortion.** We frequently use the following notation: $c_0 = f0$ and $c_n = f^n c_0$ for $n \geq 1$; for $x \in X$ and $n \geq 1$, $J(x) = |f'x|$, $J^0(x) = 1$ and $J^n(x) = J(x)J(fx)\cdots J(f^{n-1}x)$; for a compact interval $K \subset X$, $d(0, K) = \min\{|x| : x \in K\}$.

For $n \geq 1$, let

$$(5) \qquad D_n = \frac{1}{10} \cdot \left[\sum_{i=0}^{n-1} d_i^{-1}\right]^{-1} \qquad \text{where} \qquad d_i = \frac{|c_i|}{J^i(c_0)}.$$

Lemma 2.1. *For all $x, y \in I := [c_0 - D_n, c_0 + D_n] \cap X$,*

$$\frac{J^n(x)}{J^n(y)} \leq 2 \quad and \quad \left|\frac{J^n(x)}{J^n(y)} - 1\right| \leq \frac{|x - y|}{D_n}.$$

Proof. The first inequality would hold if for all $0 \leq j \leq n - 1$ we have

$$(6) \qquad 0 \notin f^j I, \quad \frac{|f^j I|}{d(0, f^j I)} \leq \log 2 \cdot d_j^{-1} \left[\sum_{i=0}^{n-1} d_i^{-1}\right]^{-1}.$$

Indeed, if this is the case, then for $x, y \in I$,

$$\log \frac{J^n(x)}{J^n(y)} \leq \sum_{j=0}^{n-1} \log \frac{J(f^j x)}{J(f^j y)} \leq \sum_{j=0}^{n-1} \frac{|f^j I|}{d(0, f^j I)} \leq \log 2.$$

The second inequality follows from $|f'x| = 2a|x|$ and $|f''x| = 2a$.

We prove (6) by induction on j. It is immediate to check (6) for $j = 0$. Let $k > 0$ and assume (6) for all $0 \leq j < k$. Summing (6) over all $0 \leq j < k$ implies $J^k(x) \leq 2J^k(y)$ for all $x, y \in I$. Hence

$$(7) \qquad |f^k I| \leq 2J^k(c_0)D_n = 2|f^k c_0|d_k^{-1}D_n \leq \frac{1}{5}|c_k|,$$

and thus $0 \notin f^k I$ holds. For the second half of (6) we have

$$\frac{|f^k I|}{d(0, f^k I)} \leq \frac{2J^k(c_0)D_n}{d(0, f^k I)} = \frac{2d_k^{-1}D_n \cdot |c_k|}{d(0, f^k I)} \leq \frac{3}{10}d_k^{-1}\left[\sum_{i=0}^{n-1} d_i^{-1}\right]^{-1}.$$

For the last inequality we have used $\frac{|c_k|}{d(0, f^k I)} \leq \frac{3}{2}$ which follows from (7).

Let K denote the subinterval of I with endpoints x, y. For $0 \leq i < n$ we have $\frac{|c_i|}{d(0, f^i K)} \leq \frac{3}{2}$. We also have $|f^i K| \leq 2J^i(c_0)|x-y|d_i d_i^{-1} = 2|x-y||c_i|d_i^{-1}$. Multiplying these two inequalities,

$$\frac{|f^i K|}{d(0, f^i K)} \leq 3|x - y|d_i^{-1}.$$

Summing this over all $0 \leq i < n$ we obtain

$$\log \frac{J^n(x)}{J^n(y)} \leq \sum_{i=0}^{n-1} \frac{|f^i K|}{d(0, f^i K)} \leq \frac{3}{10} \frac{|x-y|}{D_n} \leq \frac{3}{10}.$$

Using this and the fact that $e^z \leq 1 + 2z$ for $0 \leq z \leq \frac{3}{10}$ implies the second inequality of the lemma. $\qquad\square$

2.2. **Recovering expansion.** For $p \geq 1$, let $\gamma_p = \sqrt{e^{-\varepsilon p} D_p}$. Here, $\varepsilon > 0$ is the small constant in the statement of the proposition. The next lemma, the proof of which is a slight modification of that of [4, Lemma 1] and hence we omit, ensures an exponential growth of derivatives outside of a fixed neighborhood of 0.

Lemma 2.2. *If $f = f_a$, a is sufficiently near 2, and $x \in X$, $n \geq 1$ are such that $|f^i x| \geq \gamma_{10}$ for $i = 0, \ldots, n-1$, then $J^n(x) \geq \gamma_{10} e^{\lambda n}$. If moreover $|f^n x| < \gamma_{10}$, then $J^n(x) \geq e^{\lambda n}$.*

In what follows, fix an integer M such that $\gamma_{10} e^{\alpha \sqrt{M}} \geq 1$. Fix a_0 sufficiently near 2 so that $J(c_i) \geq 3.5$ for $0 \leq i < M$, and the conclusion of Lemma 2.2 holds for all $a \in [a_0, 2]$.

To deal with the loss of expansion due to returns to $(-\gamma_{10}, \gamma_{10})$, which is inevitable, we mimic the classical binding argument of Benedicks and Carleson [5, 6]: subdivide the interval into pieces, and deal with them independently. Key ingredients are the notion of bound periods and an associated expansion estimate. We develop them in a slightly different way from [5, 6] for our purpose.

For $p > 10$, let $I_p = [\gamma_p, \gamma_{p-1})$ and $-I_p = (-\gamma_{p-1}, \gamma_p]$. If $x \in I_p \cup -I_p$, then we regard the orbit of x as bound to the orbit of 0 up to time p, and call p the *bound period* of x.

Lemma 2.3. *If $x \in I_p \cup -I_p$, then $J^p(x) \geq e^{\frac{\lambda}{3} p}$.*

Proof. We have

$$J^p(x) = J^{p-1}(fx) \cdot J(x) \geq J^{p-1}(c_0) \cdot |x| \geq J^{p-1}(c_0) \cdot \frac{\gamma_p}{\sqrt{2}} \geq (J^{p-1}(c_0))^{\frac{2}{5}} \geq e^{\frac{\lambda}{3} p},$$

where the first inequality follows from the bounded distortion in Lemma 2.1. For the last two imequalities we have used (A2) and $p > 10$. $\qquad\square$

Lemma 2.4. *For all $0 \leq i < j$, $J^{j-i}(c_i) \geq e^{-\alpha \sqrt{i}}$.*

Proof. We start with the case where $|c_n| \geq \gamma_{10}$ holds for $n = i, \ldots, j-1$. If $j \leq M$, then the choice of a_0 ensures $J^{j-i}(c_i) \geq (3.5)^{j-i}$, which is stronger than what is asserted. If $j > M$, then using Lemma 2.2,

$$J^{j-i}(c_i) \geq \gamma_{10} e^{\lambda(j-i)} \geq \gamma_{10} e^{\alpha \sqrt{j-i}} \geq \gamma_{10} e^{\alpha \sqrt{j} - \alpha \sqrt{i}} \geq \gamma_{10} e^{\alpha \sqrt{N} - \alpha \sqrt{i}} \geq e^{-\alpha \sqrt{i}}.$$

We now consider the case where $|c_n| < \gamma_{10}$ holds for some $i \leq n < j$. Let $n_1 < n_1 + p_1 \leq n_2 < n_2 + p_2 \leq \cdots$ be defined inductively as follows: $n_1 = \min\{n \geq i : |c_n| < \gamma_{10}\}$. Given n_k with $|c_{n_k}| < \gamma_{10}$, p_k is the coresponding bound period and $n_{k+1} = \{n \geq n_k + p_k : |c_n| < \gamma_{10}\}$. For each k, the expansion estimates of Lemma 2.2 and Lemma 2.3 give

$$J^{n_{k+1} - n_k - p_k}(c_{n_k + p_k}) \geq e^{\frac{\lambda}{3}(n_{k+1} - n_k - p_k)} \quad \text{and} \quad J^{p_k}(c_{n_k}) \geq e^{\frac{\lambda}{3} p_k}.$$

Thus, if $j \in [n_k + p_k, n_{k+1}]$ for some k, then $J^{j-i}(c_i) \geq \gamma_{10} e^{\lambda(j-i)}$ holds, which is $\geq e^{-\alpha\sqrt{i}}$ as proved above. If $j \in [n_k + 1, n_k + p_k]$ for some k, then we have

$$\frac{J^j(c_0)}{J^i(c_0)} \geq \frac{J^{n_k}(c_0)}{J^i(c_0)} \cdot \frac{J^{n_k+1}(c_0)}{J^{n_k}(c_0)} \cdot \frac{J^j(c_0)}{J^{n_k+1}(c_0)} \geq e^{\frac{\lambda}{3}(n_k-i)} \cdot 2a|c_{n_k}| \cdot 2^{-1} e^{\lambda(j-n_k)}$$

$$\geq e^{\frac{\lambda}{3}(j-i)-\alpha\sqrt{n_k}} \geq e^{\frac{\lambda}{3}\sqrt{j-i}-\alpha\sqrt{n_k}} \geq e^{\frac{\lambda}{3}(\sqrt{j}-\sqrt{i})-\alpha\sqrt{j}} \geq e^{-\alpha\sqrt{i}}.$$

This completes the proof of the lemma. $\qquad\square$

We introduce a large integer N, and set $\delta = \gamma_N$. The next lemma indicates that the dynamics outside of $(-\delta, \delta)$ is uniformly expanding, with an exponent of derivatives independent of N.

Lemma 2.5. *If $x \in X$, $n \geq 1$ are such that $|f^i x| \geq \delta$ for $i = 0, 1, \ldots, n-1$, then $J^n(x) \geq \delta e^{\frac{\lambda}{3}n}$. If moreover $|f^n x| < \delta$, then $J^n(x) \geq e^{\frac{\lambda}{3}n}$.*

Proof. Let $0 \leq n_1 < n_1 + p_1 \leq n_2 < n_2 + p_2 \leq \cdots \leq n_s \leq n$ be defined inductively as follows: $n_1 = \min\{n \geq 0 \colon \delta \leq |f^n x| < \gamma_{10}\}$. Given n_k with $\delta \leq |f^{n_k} x| < \gamma_{10}$, p_k is the bound period and $n_{k+1} = \min\{n \geq n_k + p_k \colon \delta \leq |f^n x| < \gamma_{10}\}$. In view of the proof of Lemma 2.4, we have $J^{n_s}(x) \geq e^{\frac{\lambda}{3}n_s}$. If $n_s + p_s > n$, then using the bounded distortion we have $J^{n-n_s}(f^{n_s}x) \geq J(f^{n_s}x) \cdot \frac{1}{2}J^{n-n_s-1}(c_0)$, which is $\geq a\gamma_p e^{\lambda(n-n_s-1)} \geq \gamma_p e^{\frac{\lambda}{3}(n-n_s)}$. If $n_s + p_s \leq n$, then using Lemma 2.2 we have $J^{n-n_s}(f^{n_s}x) \geq \gamma_{10} e^{\frac{\lambda}{3}(n-n_s)}$. Hence, in either of the two cases we get the first estimate of the lemma.

Sublemma 2.6. *For every $n_s < i \leq n_s + p_s$, $|f^i x| \geq \delta$.*

We finish the proof of the second estimate of Lemma 2.5 assuming the conclusion of this sublemma. If $|f^n x| < \delta$, then it implies $n_s + p_s \leq n$. Hence the factor γ_{10} in the above estimate can be dropped by Lemma 2.2. Consequently we obtain the second estimate of the lemma.

To prove the sublemma, let $n_s < i \leq n_s + p_s$ and observe that the bounded distortion in Lemma 2.1 and the definition of D_{p_s} together imply $|f^i x - c_{i-n_s-1}| \leq \frac{1}{5}|c_{i-n_s-1}|$. Then

$$|f^i x| \geq \frac{4}{5}|c_{i-n_s-1}| \geq \frac{4}{5}e^{-\alpha\sqrt{p_s}} \geq \frac{4}{5}e^{-\alpha\sqrt{\frac{2}{\lambda}\log|f^{n_s}x|}} > |f^{n_s}x| \geq \delta.$$

For the second inequality we have used (A3), and Lemma 2.7 for the third. The last inequality holds because $|f^{n_s}x| < \gamma_{10} \ll 1$. $\qquad\square$

Lemma 2.7. *If $x \in I_p \cup -I_p$, then $p \leq \log|x|^{-\frac{2}{\lambda}}$. If, in addition, $p > N$, then $p \geq \log|x|^{-\frac{2}{\log 5}}$.*

Proof. We have

$$|x|^2 \leq D_{p-1} \leq \frac{1}{10}d_{p-2} \leq \frac{1}{10}J^{p-2}(c_0)^{-1} \leq \frac{1}{10}e^{-\lambda(p-2)} \leq e^{-\lambda p}.$$

The fourth inequality follows from (A2). Rearranging this yields the upper estimate of p. On the other hand we have

$$|x|^2 \geq e^{-\varepsilon p}D_p \geq \frac{e^{-\varepsilon p}}{p}\min_{0 \leq i < p} d_i \geq \frac{e^{-\varepsilon p}}{p}\frac{e^{-\alpha\sqrt{p}}}{4^p} \geq e^{-2\varepsilon p}4^{-p},$$

where the last inequality holds for sufficiently large N. Rearranging this yields the lower estimate of p. $\qquad\square$

2.3. **Estimates on basic pieces of intervals.** For each $p > N$, cut I_p into $[e^{3\varepsilon p}]$-number of intervals of equal length and denote them by $I_{p,j}$ $(j = 1, 2, \cdots)$. Let $I_{p,-j} = -I_{p,j}$.

Lemma 2.8. *The following estimates hold:*

(a) $|f^p I_{p,j}| \geq e^{-5\varepsilon p}$;
(b) $|I_{p,j}| \leq d(0, I_{p,j})^{1+\frac{\varepsilon}{3}}$;
(c) *for all* $x, y \in I_{p,j}$, $\log \frac{J^p(x)}{J^p(y)} \leq |f^p x - f^p y|^{\varepsilon^2}$.

Proof. As for (a) we have

$$|f I_{p,j}| \geq e^{-3\varepsilon p} |\gamma_{p-1} - \gamma_p| \cdot \gamma_p = e^{-4\varepsilon p} \left(e^{\frac{\varepsilon}{2}} - 1 \right) D_p,$$

and thus

$$|f^p I_{p,j}| \geq \frac{1}{2} J^{p-1}(c_0) |f I_{p,j}| \geq e^{-4\varepsilon p} \left(e^{\frac{\varepsilon}{2}} - 1 \right) \cdot D_p J^{p-1}(c_0).$$

To estimate the second factor, using (A3) and Lemma 2.4 we have

$$(8) \qquad J^{p-1}(c_0)^{-1} D_p^{-1} = \sum_{j=0}^{p-1} |c_j|^{-1} \frac{J^j(c_0)}{J^{p-1}(c_0)} \leq \sum_{j=0}^{p-1} e^{2\alpha\sqrt{j}} \leq e^{3\alpha\sqrt{p}}.$$

Taking reciprocals and plugging the result into the above inequality,

$$|f^p I_{p,j}| \geq (e^{\frac{\varepsilon}{2}} - 1) e^{-4\varepsilon p - 3\alpha\sqrt{p}} \geq e^{-5\varepsilon p}.$$

The last inequality holds for sufficiently large N.

As for (b) we have

$$(9) \qquad \frac{\gamma_{p-1}^2}{\gamma_p^2} \leq e^{\varepsilon} \left(1 + \frac{d_{p-1}}{d_p} \right) = e^{\varepsilon} \left(1 + \frac{|c_{p-1}|}{J(c_{p-1})|c_p|} \right) \leq 3 e^{\alpha\sqrt{p}},$$

where the last inequality follows from (A3). Hence $|I_p| \leq \gamma_{p-1} \leq \sqrt{3 e^{\alpha\sqrt{p}}} \cdot \gamma_p$, and thus $|I_{p,j}| \leq e^{-3\varepsilon p} |I_p| \leq e^{-\varepsilon p} \gamma_p$, where the last inequality holds if N is sufficiently large. Since $e^{-\varepsilon p} \leq 10^{-\frac{\varepsilon p}{3}} \leq \gamma_p^{\frac{\varepsilon}{3}}$ we have $|I_{p,j}| \leq e^{-\varepsilon p} \gamma_p \leq \gamma_p^{1+\frac{\varepsilon}{3}} \leq d(0, I_{p,j})^{1+\frac{\varepsilon}{3}}$.

It is left to prove (c). We have $|J^p(x) - J^p(y)| \leq I + II$, where

$$I = J^{p-1}(fx) \cdot |J(x) - J(y)|, \quad II = J(y) \cdot |J^{p-1}(fx) - J^{p-1}(fy)|.$$

Using (b),

$$(10) \qquad I \leq 8 J^{p-1}(c_0)| \cdot |x - y| \leq 8 J^{p-1}(c_0) \cdot d(0, I_{p,j}) |x - y|^{\frac{\varepsilon}{3+\varepsilon}}.$$

Lemma 2.1 gives

$$\left| \frac{J^{p-1}(fx)}{J^{p-1}(fy)} - 1 \right| \leq \frac{|x - y|^2}{D_{p-1}} \leq \frac{|x - y|^2}{d(0, I_{p,j})^2} \leq |x - y|^{\frac{2\varepsilon}{3+\varepsilon}}.$$

(b) implies $J(y) \leq 2 d(0, I_{p,j})$, and thus

$$(11) \qquad II \leq J(y) J^{p-1}(fy) \left| \frac{J^{p-1}(fx)}{J^{p-1}(fy)} - 1 \right| \leq 4 d(0, I_{p,j}) |J^{p-1}(c_0)| |x - y|^{\frac{2\varepsilon}{3+\varepsilon}}.$$

Combining (10) and (11) with $J^p(y) \geq J^{p-1}(c_0) \cdot d(0, I_{p,j})$ yields

$$\left| \frac{J^p(x)}{J^p(y)} - 1 \right| \leq 8 |x - y|^{\frac{\varepsilon}{3+\varepsilon}} \leq |f^p x - f^p y|^{\frac{\varepsilon}{3+\varepsilon}} \leq |f^p x - f^p y|^{\varepsilon^2},$$

which implies (c). The second inequality holds for sufficiently large N. The last inequality is because $|f^p x - f^p y| < 1$ and $\frac{\varepsilon}{3+\varepsilon} > \varepsilon^2$. $\qquad\square$

2.4. **Return time estimate.** Let Λ^+ be the right extremal $I_{p,j}$-interval. Lemma 2.8(b) implies $0 \notin 3\Lambda^+$ ($3\Lambda^+ :=$ the interval centered at the midpoint of Λ^+ and three times its length). Let $\Lambda^- = -\Lambda^+$ and $\Lambda = \Lambda^- \cup \Lambda^+$. We will sometimes refer to intervals of the form $[\delta, \delta + l)$ or $(-\delta - l, -\delta]$ with $l \geq |\Lambda^+|$ as $I_{p,j}$-intervals with $p = 0$. The next lemma is an extension of the classical return time estimate [6] to generic intervals.

Lemma 2.9. *Let $\omega_0 = f^q I_{p,j}$, where $q = 0$ or $p \leq q \leq \min\{k \geq p : f^k I_{p,j} \cap (-\delta, \delta) \neq \emptyset\}$. There exist a countable partition $\mathcal{Q}$ of ω_0 into closed intervals and a stopping time function $S \colon \mathcal{Q} \to \mathbb{N}$ such that:*

(a) *for each $\omega \in \mathcal{Q}$, $f^{S(\omega)}$ sends ω diffeomorphically onto Λ^+ or Λ^-;*
(b) *for each $\omega \in \mathcal{Q}$ and $x, y \in \omega$, $\log \frac{J^{S(\omega)}(x)}{J^{S(\omega)}(y)} \leq 10\delta^{-3}|f^{S(\omega)}x - f^{S(\omega)}y|^{\varepsilon^2}$ and $\frac{J^{S(\omega)}(x)}{J^{S(\omega)}(y)} \leq 2$;*
(c) *for every $n \geq 40 \cdot \max(p, N)$, $|\cup \{\omega \in \mathcal{Q} \colon S(\omega) \geq n\}| \leq e^{-\frac{\lambda}{9}n}|\omega_0|$;*
(d) *for every $n \geq \frac{3}{\varepsilon}$, $\#\{\omega \in \mathcal{Q} \colon S(\omega) = n\} \leq e^{5\varepsilon n}$.*

Proof. We first define and describe the combinatorics of the partition $\mathcal{Q}$ and the stopping time S. We construct a sequence $\tilde{\mathcal{P}}_n$ ($n = 0, 1, \cdots$) of partitions of ω_0 inductively as follows. If $p \neq 0$ and $q = 0$, then let $p_0 = p$. In all other cases, let $p_0 = 0$. Define $\tilde{\mathcal{P}}_n = \{\omega_0\}$ for $0 \leq n < \min\{n \geq p_0 \colon f^n \omega \cap (-\delta, \delta) \neq \emptyset\}$.

The f^n-images of elements of $\tilde{\mathcal{P}}_{n-1}$ are in two phases: either *bound* or *free*. If $\omega \in \tilde{\mathcal{P}}_n$, $f^n \omega$ is free and intersects $(-\delta, \delta)$, then an integer is attached, which is called a *bound period* and is denoted by $p(f^n \omega)$. We say $f^n \omega$ is *bound* if there exists $k < n$ and $\omega \in \tilde{\mathcal{P}}_k$ to which a bound period $p(f^k \omega)$ is attached and satisfies $n < k + p(f^k \omega)$. Otherwise we say $f^n \omega$ is *free*.

Given $\omega \in \tilde{\mathcal{P}}_{n-1}$, $\tilde{\mathcal{P}}_n|\omega$ is defined as follows. If $f^n \omega$ is free and contains at least two $I_{p,j}$'s, then $\tilde{\mathcal{P}}_n$ partitions ω according to the (p, j)-locations of its f^n-images. Partition points are inserted only to ensure that the f^n-images of $\tilde{\mathcal{P}}_n$-elements contain exactly one $I_{p,j}$ and do not intersect more than three $I_{p,j}$'s. In all other cases, let $\tilde{\mathcal{P}}_n|\omega = \{\omega\}$.

If $\tilde{\mathcal{P}}_n$ partitions ω, bound periods $p(\cdot)$ for the f^n-images of the elements of $\tilde{\mathcal{P}}_n|\omega$ are defined by their p-locations. Otherwise, define $p(f^n \omega) = \min\{p \colon (I_p \cup -I_p) \cap f^n \omega \neq \emptyset\}$ only if $f^n \omega$ is free and intersects Γ_N.

The stopping time S and the partition $\mathcal{Q}$ are defined as follows. Let $\omega \in \tilde{\mathcal{P}}_{n-1}$. If $f^n \omega$ is free and $f^n \omega \supset 3\Lambda^+$ or $3\Lambda^-$, then set $\omega \cap f^{-n}\Lambda^+ \in \mathcal{Q}$ or $\omega \cap f^{-n}\Lambda^- \in \mathcal{Q}$, and $S(\omega) = n$. We iterate the remaining parts $f^n \omega \setminus \Lambda^+$ or $f^n \omega \setminus \Lambda^-$, which is the union of elements of $\tilde{\mathcal{P}}_n$, and repeat the same procedure. (a) is obvious. (b) follows from the next

Lemma 2.10. *If $\omega \in \tilde{\mathcal{P}}_{n-1}$ and $f^n \omega$ is free, then for all $x, y \in \omega$ we have*

$$\log \frac{J^n(x)}{J^n(y)} \leq 10\delta^{-3}|f^n x - f^n y|^{\varepsilon^2}.$$

If moreover $f^n \omega \subset (-2\delta, 2\delta)$, then for all $x, y \in \omega$ we have

$$\log \frac{J^n(x)}{J^n(y)} \leq 2 \sum_{k=1}^{\infty} e^{-\varepsilon^2 \frac{\lambda}{3} k}.$$

Proof. Let $n_1 < \cdots < n_s < n$ denote all the free returns in the first n-iterates of ω, with $p_1, \ldots, p_s$ the corresponding bound periods as defined above. For iterates inside bound periods,

$$(12) \qquad \sum_{j=1}^{s} \log \frac{J^{p_j}(f^{n_j}x)}{J^{p_j}(f^{n_j}y)} \le \sum_{j=1}^{s} |f^{n_j+p_j}x - f^{n_j+p_j}y|^{\varepsilon^2} \le \delta^{-\varepsilon^2} \sum_{k=1}^{\infty} e^{-\varepsilon^2 \frac{\lambda}{3}k} \cdot |f^n x - f^n y|^{\varepsilon^2}.$$

For the first inequality we have used Lemma 2.8(c). For the second we have used $|f^{n_j+p_j}x - f^{n_j+p_j}y| \le \delta^{-1} e^{-\frac{\lambda}{3}(n-n_j-p_j)}|f^n x - f^n y|$, which follows from decomposing the time interval $[n_j + p_j, n]$ into bound and free segments, and then applying Lemma 2.3 to each bound segment and Lemma 2.5 to each free segment. For iterates out of bound periods, letting $n_0 + p_0 = 0$ and $n_{s+1} = n$ we have

$$(13) \qquad \sum_{j=0}^{s} \sum_{i=n_j+p_j}^{n_{j+1}-1} \log \frac{J(f^i x)}{J(f^i y)} \le 4\delta^{-1} \sum_{j=0}^{s} \sum_{i=n_j+p_j}^{n_{j+1}-1} |f^i x - f^i y| \le 4\delta^{-2} \sum_{k=1}^{\infty} e^{-\frac{\lambda}{3}k} \cdot |f^n x - f^n y|.$$

(12) (13) yield the first inequality. If $f^n \omega \subset (-2\delta, 2\delta)$, then δ^{-2} in (13) may be replaced by δ^{-1}, by virtue of the last assertion of Lemma 2.5. Hence the second assertion holds. $\square$

We prove (c). To each $\omega \in \mathcal{Q}$ with $S(\omega) \ge n$ we associate an *itinerary*

$$(n_1, p_1, j_1), \ldots, (n_s, p_s, j_s)$$

up to time n as follows: $0 < n_1 < \cdots < n_s < n$ is a sequence of integers, associated with a sequence $\omega_0 \supset \omega_{n_1} \supset \cdots \supset \omega_{n_s} \supset \omega$ of intervals such that for each $1 \le i \le s$, ω_{n_i} is the element of $\tilde{\mathcal{P}}_{n_i}$ containing ω that arises out of the subdivision at time n_i, with $f^{n_i}\omega_{n_i} \cap (-\delta, \delta) \ne \emptyset$, with (p_i, j_i) being its (p, j)-location.

Let $\mathcal{Q}'_{\ge n}$ denote the collection of all $\omega \in \mathcal{Q}$ such that $S(\omega) \ge n$, and there exists $0 < k < n$ such that $f^k \omega_k \cap (-\delta, \delta) \ne \emptyset$ holds for the element ω_k of $\tilde{\mathcal{P}}_k$ containing ω. Let $\mathcal{Q}''_{\ge n}$ denote the collection of all $\omega \in \mathcal{Q}$ such that $S(\omega) \ge n$ and $\omega \notin \mathcal{Q}'_{\ge n}$. We have $\{\omega \in \mathcal{Q}: S(\omega) \ge n\} = \mathcal{Q}'_{\ge n} \cup \mathcal{Q}''_{\ge n}$. We estimate contributions from each separately.

Let $\omega \in \mathcal{Q}'_{\ge n}$. Let $n_{s+1} \ge n$ denote the integer such that $\tilde{\mathcal{P}}_{n_{s+1}}$ partitions ω_{n_s}. Since $J^{n_{s+1}}(x) \ge \delta e^{\frac{\lambda}{3}\sum_{i=0}^{s} p_i}$ for any $x \in \omega$ we have

$$(14) \qquad\qquad\qquad |\omega| \le \delta^{-1} e^{-\frac{\lambda}{3}\sum_{i=0}^{s} p_i}.$$

We count the number of partition elements with a given combinatorics. Lemma 2.7 gives $p_i \ge \log \delta^{-\frac{2}{\log 5}}$. By Stirling's formula for factorials and $1 \le |j_i| \le e^{3\varepsilon p_j}$, for a given integer $R > 0$ the number of all feasible $(p_i, j_i)_{i=1}^{s}$ with $\sum_{i=1}^{s} p_i = R$ is $\le 2^s \binom{R+s}{s} e^{3\varepsilon R} \le e^{\beta(N)+3\varepsilon R}$, where $\beta(N) \to 0$ as $N \to \infty$.

To estimate R from below, we claim that

$$n_{i+1} - n_i - p_i \le \frac{6\varepsilon p_i}{\lambda} - \log \delta \le p_i, \quad 1 \le \forall i \le s.$$

Indeed, Lemma 2.8(a) gives $|f^{n_i+p_i}\omega_i| \ge e^{-5\varepsilon p_i}$, and if the first inequality were false, then $|f^{n_{i+1}}\omega_i| > 2$, a contradiction because $|X| = 2$. The second inequality follows from Lemma 2.7. Summing this estimate over all $1 \le i \le s$ gives $n < n_{s+1} \le 2\sum_{i=1}^{s} p_i$, and thus $R \ge n/2$.

Since there are at most $\binom{n}{s}$ number of ways of distributing $n_1, \ldots, n_s$ in $(0, n)$, summing (14) over all $\omega \in \mathcal{Q}'_{\geq n}$ we obtain

$$(15) \qquad \sum_{\omega \in \mathcal{Q}'_{\geq n}} |\omega| = \sum_{R,s} \sum_{\substack{(n_i, p_i, j_i) \\ \sum_{i=1}^{s} p_i = R}} |\omega| \leq \delta^{-1} e^{-\frac{\lambda}{3} p_0} \sum_{s} \sum_{R \geq n/2} \binom{n}{s} e^{\beta(N) - (\frac{\lambda}{3} - 3\varepsilon)R} \leq \delta^{-1} e^{-\frac{\lambda}{7} n}.$$

To estimate contributions from elements in $\mathcal{Q}''_{\geq n}$ we need a slightly different argument, because they have no itinerary. for each $k \leq n$ set $\mathcal{Q}_k = \{\omega \in \mathcal{Q} \colon S(\omega) = k\}$. Let $\mathcal{A}_k$ denote the collection of all $\omega \in \tilde{\mathcal{P}}_{k-1}$ such that there exists $0 < j < k$ such that $f^j \omega_j \cap (-\delta, \delta) \neq \emptyset$ holds for the element ω_j of $\tilde{\mathcal{P}}_j$ containing ω. Elements of $\mathcal{A}_k$ have itineraries, and the sum of all bound periods up to time k is obviously $\leq k$. In view of this and the above counting argument, we have $\#\mathcal{A}_k \leq e^{4\varepsilon k}$.

By construction, for each $\omega \in \mathcal{Q}''_{\geq n}$ there are two mutually exclusive possibilities:

- there exist $k < n$ and $\omega' \in \mathcal{A}_k$ such that $\omega \cup \omega'$ is an interval, contained in the same element of $\tilde{\mathcal{P}}_{k-1}$ which contains at most two elements of $\mathcal{Q}_n$;
- $\omega_{n-1} \in \tilde{\mathcal{P}}_m$, where m is the integer such that $\tilde{\mathcal{P}}_m$ partitions ω_0.

The second possibility is eliminated by Lemma 2.5 and the assumption on the range of n. Hence we have

$$\#\mathcal{Q}''_{\geq n} \leq 2 \sum_{k=1}^{n} \#\mathcal{A}_k \leq 2 \sum_{k=1}^{n} e^{4\varepsilon k} \leq e^{5\varepsilon n}.$$

For each $\omega \in \mathcal{Q}''_{\geq n}$ we have $|\omega| \leq \delta^{-1} e^{-\frac{\lambda}{3} n}$. Hence

$$(16) \qquad \sum_{\omega \in \mathcal{Q}''_{\geq n}} |\omega| \leq \#\mathcal{Q}''_{\geq n} \cdot \delta^{-1} e^{-\frac{\lambda}{3} n} \leq \delta^{-1} e^{-\frac{\lambda}{4} n}.$$

(15) (16) yield

$$\sum_{\omega \in \mathcal{Q} \colon S(\omega) \geq n} |\omega| = \sum_{\omega \in \mathcal{Q}'_{\geq n}} |\omega| + \sum_{\omega \in \mathcal{Q}''_{\geq n}} |\omega| \leq \delta^{-1} e^{-\frac{\lambda}{8} n}.$$

Observe that $|\omega_0| \geq \delta(1 - e^{-\frac{\beta}{2}}) 5^{-\max(p, N)}$. This implies $e^{-\frac{\lambda}{8} n} \leq e^{-\frac{\lambda}{9} n} |\omega_0|$ provided $n \geq 40 \cdot \max(p, N)$. This completes the proof of (c).

As for (d), let $\omega_{n-1} \in \tilde{\mathcal{P}}_{n-1}$ and suppose that ω_{n-1} contains an element of $\mathcal{Q}_n$. There are three mutually exclusive possibilities:

- $\omega_{n-1} \in \mathcal{A}_n$;
- there exist $k < n$ and $\omega' \in \mathcal{A}_k$ such that $\omega \subset \omega'$ and ω' contains at most two elements of $\mathcal{Q}_n$;
- $\omega_{n-1} \in \tilde{\mathcal{P}}_m$.

Each such ω_{n-1} contains at most two elements of $\mathcal{Q}_n$, and there are at most two elements corresponding to the last possibility. Hence

$$\#\mathcal{Q}_n \leq 4 + 2 \sum_{k=1}^{n} \#\mathcal{A}_k \leq e^{5\varepsilon n}.$$

This completes the proof of (d) and hence that of Lemma 2.9. $\qquad \square$

Notation. In what follows we apply Lemma 2.9 to different intervals. Unless otherwise stated, the corresponding partitions and stopping times will be simply denoted by $\mathcal{Q}$ and S. We denote by $\{S \geq n\}$ the union of $\omega \in \mathcal{Q}$ with $S(\omega) \geq n$. The meaning of $\{S < n\}$ is analogous.

3. Proof of the proposition

In this section we prove the proposition. We construct an induced map and an associated tower parametrized by ε, and use them to find a horseshoe which carries an ergodic measure σ as in the statement of the proposition.

3.1. Construction of a positive measure set.
We construct a subset Ω_∞^+ of Λ^+ with positive Lebesgue measure. With an abuse of notation, for the rest of this paper let $\tilde{\mathcal{P}}_n$ ($n = 0, 1, \cdots$) denote the sequence of partitions of Λ^+ or Λ^- obtained by performing the construction in the proof of Lemma 2.9 for Λ^+. Let $\Omega_0 = \Lambda^+$, and let $\Omega_0 = \Omega_1 = \cdots = \Omega_{N-1}$. For $n \geq N$, define inductively

$$\Omega_n = \Omega_{n-1} \setminus \bigcup \{\omega \in \tilde{\mathcal{P}}_n \colon f^n \omega \cap (-\delta e^{-\varepsilon n}, \delta e^{-\varepsilon n}) \neq \emptyset\},$$

and set $\Omega_\infty^+ = \bigcap_{n \geq 0} \Omega_n$. Any component of $\Omega_{n-1} \setminus \Omega_n$ is called a *gap of order n*.

Lemma 3.1. $|\Omega_\infty^+| \geq \frac{1}{2}|\Lambda^+|$.

Proof. Let $n \geq N$. Let $\omega \in \tilde{\mathcal{P}}_{n-1}$ be such that $\omega \subset \Omega_{n-1}$, and suppose that some part of it will be deleted at step n. We claim that $f^n \omega$ is free. To see this, suppose the contrary and let k denote the last free return time of ω with bound period p. We have $n < k + p$. We have $|f^{k+1} \omega| \leq 2\gamma_p^2$, and thus $|f^n \omega| \leq J^{n-k-1}(c_0) D_p \leq \frac{1}{10}|c_{n-k}|$. Hence

$$(17) \qquad d(0, f^n \omega) \geq \frac{9}{10}|c_{n-k}| \geq \frac{9}{10}e^{-\alpha\sqrt{p}} \geq \frac{9}{10}e^{-\alpha\sqrt{\frac{2}{\lambda}(-\log \delta + \varepsilon n)}} > \frac{\delta}{2}e^{-\varepsilon n}.$$

The second inequality follows from (A3). Hence no part of ω is deleted at step n and a contradiction arises.

Since $f^k \omega \supset$ some $I_{p,j}$ with $p \leq \frac{2}{\lambda}(-\log \delta + \varepsilon n)$, we are guaranteed that

$$|f^n \omega \cap (-\delta, \delta)| \geq \delta \cdot \min\left(e^{-6\varepsilon^2 n}, 1 - e^{-\varepsilon n}\right) \geq \delta e^{-6\varepsilon^2 n}.$$

But the subinterval of $f^n \omega$ to be deleted has length $\leq 3\delta e^{-\varepsilon n}$. Taking distortion into consideration when pulling back to Ω_0, we have

$$\frac{|\Omega_{n-1} - \Omega_n|}{|\Omega_{n-1}|} \leq e^{2 \sum_{k=1}^\infty e^{-\frac{\varepsilon^2 \lambda k}{3}} - (\varepsilon - 6\varepsilon^2)n}.$$

The statement of the lemma follows since

$$\frac{|\Omega_\infty^+|}{|\Lambda^+|} \geq \prod_{n=N}^\infty \left(1 - e^{2 \sum_{k=1}^\infty e^{-\frac{\varepsilon^2 \lambda k}{3}} - (\varepsilon - 6\varepsilon^2)n}\right) \geq \frac{1}{2},$$

where the last inequality holds for sufficiently large N. $\qquad\square$

3.2. **Construction of an induced map on the Cantor set.** Let $\Omega_\infty^- = -\Omega_\infty^+$ and let $\Omega_\infty = \Omega_\infty^+ \cup \Omega_\infty^-$. We construct an induced map $F \colon \Omega_\infty \circlearrowleft$. With an abuse of notation, let $\mathcal{Q}$ denote the partition and S the stopping time obtained with Lemma 2.9 applied to Λ^+.

Definition 3.2. Let $x \in \Omega_\infty$, $n > 0$ be such that $f^n x \in \Lambda$. We say n is a *regular return time* of x if:

- let ω denote the element of $\tilde{\mathcal{P}}_{n-1}$ containing x. Then $f^n \omega$ is free;
- $3\Lambda^+ \subset f^n \omega$ or $3\Lambda^- \subset f^n \omega$.

Let $x \in \Omega_\infty$. We define inductively a sequence $R_1(x) < R_2(x) < \cdots$ of regular return times of x and a return time $R(x)$ to Ω_∞ as follows. Start with $R_1(x) = S(x)$. Given $R_i(x)$, there are two cases: either (i) $f^{R_i(x)} x \in \Omega_\infty$, or (ii) $f^{R_i(x)} x \notin \Omega_\infty$. In case (i), let $R(x) = R_i(x)$. In case (ii), let g_i denote the order of the gap containing $f^{R_i(x)} x$ and define $R_{i+1}(x)$ to be the next regular return time after $R_i + g_i$.

Lemma 3.3. *There exists a countable partition $\mathcal{Q}^+$ of a full measure subset of Ω_∞^+ such that the following holds for every $\omega \in \mathcal{Q}^+$:*

   (a) *R is constant on ω and $f^{R(\omega)} \omega = \Omega_\infty^+$ or $= \Omega_\infty^-$;*
   (b) *for all $x \in \widehat{\omega}$, $J^{R(\omega)}(x) \geq e^{\frac{\lambda}{3} R(\omega)}$. Here, $\widehat{\omega}$ is the smallest closed interval containing ω;*
   (c) *for any $x, y \in \widehat{\omega}$, $\log \frac{J^{R(\omega)}(x)}{J^{R(\omega)}(y)} \leq 10\delta^{-1} |f^{R(\omega)} x - f^{R(\omega)} y|^{\varepsilon^2}$;*
   (d) *for every $n \geq N$, $|\cup \{\omega \in \mathcal{Q}^+ \colon R(\omega) \geq n\}| \leq \delta^{-1} e^{-\frac{\lambda}{10} n}$;*
   (e) *for every $n \geq N$, $\#\{\omega \in \mathcal{Q}^+ \colon R(\omega) = n\} \leq e^{6\varepsilon n}$.*

Let $\mathcal{Q}^-$ denote the partition of Ω_∞^- which is the mirror image of $\mathcal{Q}^+$ with respect to 0. Let $\mathcal{Q}' = \mathcal{Q}^+ \cup \mathcal{Q}^-$. Extend the return time function $R \colon \mathcal{Q}^+ \to \mathbb{N}$ to $\mathcal{Q}'$ in the obvious way. Define an induced map $F \colon \Omega_\infty \circlearrowleft$ by $F|\omega = f^{R(\omega)}$ for $\omega \in \mathcal{Q}'$.

Proof of Lemma 3.3. We construct $\mathcal{Q}^+$ by induction. For all $\omega \in \mathcal{Q}$ which intersects Ω_∞^+, $(f^{S(\omega)}|\omega)^{-1} (f^{S(\omega)} \omega \cap \Omega_\infty)$ consists of at most two components. We let them in $\mathcal{Q}^+$. These are indeed subsets of Ω_∞^+, by the next

Sublemma 3.4. *If $\omega \in \mathcal{Q}$ and $\omega \cap \Omega_\infty^+ \neq \emptyset$, then $(f^{S(\omega)}|\omega)^{-1} (f^{S(\omega)} \omega \cap \Omega_\infty) \subset \Omega_\infty^+$.*

Proof. Let $x \in (f^{S(\omega)}|\omega)^{-1} (f^{S(\omega)} \omega \cap \Omega_\infty)$. Since $\omega \cap \Omega_\infty^+ \neq \emptyset$, $\omega \subset \Omega_{S(\omega)-1}$ holds, and thus $x \in \Omega_{S(\omega)-1}$. Since $f^{S(\omega)} x \in \Omega_\infty$, $x \in \Omega_{S(\omega)}$ holds. Since $f^{S(\omega)} x = y$ for some $y \in \Omega_\infty$, for every $n > 0$ we have $|f^{n+S(\omega)} x| = |f^n y| \geq \delta e^{-\beta n} > \delta e^{-\beta(n+S(\omega))}$, and thus $x \in \Omega_{n+S(\omega)}$. This yields $x \in \Omega_\infty^+$. $\qquad\square$

Proceeding to the next step, let $\omega \in \mathcal{Q}$ intersect Ω_∞^+. Let G be a gap with $G \cap f^{S(\omega)}(\omega \cap \Omega_\infty^+) \neq \emptyset$. Let g denote the order of G. If there exist $k > S(\omega) + g$ and $\omega' \in \tilde{\mathcal{P}}_{k-1}$ such that: (i) $f^{S(\omega)} \omega' \subset G$; (ii) there exists $x \in \omega'$ such that k is the next regular return time of x after $S(\omega)$; (iii) $\omega' \cap \Omega_\infty^+ \neq \emptyset$, then we let $(f^k|\omega')^{-1} (f^k \omega' \cap \Omega_\infty) \in \mathcal{Q}'$. This is indeed a subset of Ω_∞^+, by the next

Sublemma 3.5. *$(f^k|\omega')^{-1} (f^k \omega' \cap \Omega_\infty) \subset \Omega_\infty^+$.*

Proof. Let $x \in (f^k|\omega')^{-1} (f^k \omega' \cap \Omega_\infty)$. Since $\omega' \cap \Omega_\infty^+ \neq \emptyset$, $x \in \Omega_{k-1}$ holds. Since $f^k x \in \Omega_\infty$, $x \in \Omega_k$ holds. Arguing similarly to the proof of Sublemma 3.4, for every $n > 0$ we have $|f^{n+k} x| \geq \delta e^{-\beta n} > \delta e^{-\beta(n+k)}$, and thus $x \in \Omega_{n+k}$. This yields $x \in \Omega_\infty^+$. $\qquad\square$

In generic steps we treat points send into gaps in the previous steps. This completes the inductive construction of $\mathcal{Q}^+$. (a) (b) are obvious. (c) follows from Lemma 2.10.

To prove (d) (e) we need some preliminary considerations. Let $\{R \geq n\}$ denote the union of all $\omega' \in \mathcal{Q}'$ with $R(\omega') \geq n$. For $0 \leq k \leq n$, let

$$\mathcal{P}'_k = \{\omega \in \tilde{\mathcal{P}}_k \colon \omega \cap \{R \geq n\} \neq \emptyset\}.$$

Let $\omega \in \mathcal{P}'_k$. By construction, all points in ω share the same history up to time k, and thus share the same sequence of regular return times up to time k, which we denote by $0 =: R_0 < R_1(\omega) < R_2(\omega) < \cdots \leq k$. Let $\omega_0 = \Lambda^+$, and for $i \geq 1$ let ω_i denote the element of $\tilde{\mathcal{P}}_{R_i}$ which contains ω. The next two estimates will be used in what follows:

$$(18) \qquad\qquad R_{i+1} - R_i \geq N \quad\text{and}\quad \frac{|\omega_{i+1}|}{|\omega_i|} \leq 2e^{-\frac{\lambda}{3}(R_{i+1}-R_i)}.$$

The first estimate follows from $p(f^{R_i}\omega_i) \geq N$ and that $f^{R_{i+1}}\omega_i$ is free. As for the second one, the mean value theorem gives $|f^{R_i}\omega_i| = J^{R_i}(x)|\omega_i|$ and $|f^{R_i}\omega_{i+1}| = J^{R_i}(y)|\omega_{i+1}|$ for some $x \in \omega_i$ and $y \in \omega_{i+1}$. Since $f^{R_i}\omega_i = \Lambda^+$ or $= \Lambda^-$, Lemma 2.8(b) and the proof of Lemma 2.10 together imply $J^{R_i}(x) \leq 2J^{R_i}(y)$. Hence we have $\frac{|\omega_{i+1}|}{|\omega_i|} \leq 2\frac{|f^{R_i}\omega_{i+1}|}{|f^{R_i}\omega_i|} = 2\frac{|f^{R_i}\omega_{i+1}|}{|\Lambda^+|}$. Meanwhile we have $e^{\frac{\lambda}{3}(R_{i+1}-R_i)}|f^{R_i}\omega_{i+1}| \leq |f^{R_{i+1}-R_i}(f^{r_i}\omega_{i+1})| = |\Lambda^+|$. Rearranging this and plugging the result into the right-hand-side gives the second estimate.

Let $n \geq N$. For $1 \leq p \leq \frac{n}{N}$, let

$$Q_n^p = \cup\{\omega \in \mathcal{P}'_n \colon \text{the number of regular return times up to time } n \text{ is } p\}.$$

Observe that $|\{R \geq n\}| \leq \sum_{p=1}^{\frac{n}{N}} |Q_n^p|$. To estimate $|Q_n^p|$ we argue as follows. Let $1 \leq i \leq p$. For an i-string $(k_1, \cdots, k_i)$ of positive integers, let

$$\mathcal{Q}_n(k_1, \ldots, k_i) = \left\{\omega \in \mathcal{P}'_{k_1+k_2+\cdots+k_i} \colon R_j(\omega) = k_1 + k_2 + \cdots + k_j \;\; 1 \leq \forall j \leq i\right\},$$

and write $|\mathcal{Q}_n(k_1, \ldots, k_p)| = \sum_{\omega \in \mathcal{Q}_n(k_1,\ldots,k_p)} |\omega|$.

Sublemma 3.6. $|\mathcal{Q}_n(k_1, \ldots, k_p)| \leq e^{-\frac{\lambda}{4}(k_1+\cdots+k_p)}$.

Proof. Let $1 \leq i < p$. For each $\omega_i \in \mathcal{Q}_n(k_1, \ldots, k_i)$, let

$$\mathcal{Q}(\omega_i, k_{i+1}) = \{\omega_{i+1} \in \mathcal{Q}_n(k_1, \ldots, k_{i+1}) \colon \omega_{i+1} \subset \omega_i\}.$$

Then

$$(19) \qquad |\mathcal{Q}(k_1, \ldots, k_{i+1})| = \sum_{\omega_i \in \mathcal{Q}(k_1,\ldots,k_i)} |\omega_i| \sum_{\omega_{i+1} \in \mathcal{Q}(\omega_i, k_{i+1})} \frac{|\omega_{i+1}|}{|\omega_i|}.$$

For the sum of the fractions, using $\#\mathcal{Q}_n(\omega_i, k_{i+1}) \leq e^{5\varepsilon k_{i+1}}$ from Lemma 2.9(d) and the second estimate of (18),

$$\sum_{\omega_{i+1} \in \mathcal{Q}(\omega_i, k_{i+1})} \frac{|\omega_{i+1}|}{|\omega_i|} \leq \#\mathcal{Q}_n(\omega_i, k_{i+1}) \cdot 2e^{-\frac{\lambda}{3}k_{i+1}} \leq e^{-\frac{\lambda}{4}k_{i+1}}.$$

Plugging this into the right-hand-side of (19) we get $|\mathcal{Q}_n(k_1, \ldots, k_{i+1})| \leq e^{-\frac{\lambda}{4}k_{i+1}}|\mathcal{Q}_n(k_1, \ldots, k_i)|$. Using this inductively and $|\mathcal{Q}_n(k_1)| \leq e^{-\frac{\lambda}{4}k_1}$, and then substituting $i = p - 1$ we obtain the desired inequality. $\qquad\square$

Sublemma 3.7. *If $k_1 + \cdots + k_p \leq \frac{n}{2}$, then $|\mathcal{Q}_n(k_1, \ldots, k_p)| \leq \delta^{-1} e^{-\frac{\lambda}{5}n}$.*

Proof. Let $\omega \in \mathcal{Q}_n(k_1, \ldots, k_p)$. If $f^n\omega$ is free, then $|\omega| \leq \delta^{-1} e^{-\frac{\lambda}{3}n}$. If $f^n\omega$ is bound, then let $k < n$ denote the free return with bound period p with $k < n < k + p$. Since ω intersects Ω_∞, $d(0, f^k\omega) \geq \frac{1}{2}\delta e^{-\varepsilon n}$. Hence $k + p - n \leq \frac{3\varepsilon}{\lambda}n$, and thus for all $x \in \omega$,

$$J^n(x) = J^{k+p}(x)/J^{k+p-n}(f^n x) \geq 4^{-(k+p-n)} e^{\frac{\lambda}{3}(k+p)} \geq 4^{-\frac{3\varepsilon}{\lambda}n} e^{\frac{\lambda}{3}n} \geq e^{\frac{\lambda}{4}n}.$$

Hence $|\omega| \leq e^{-\frac{\lambda}{4}n}$ holds. On the other hand, Lemma 2.9(c) implies $\#\mathcal{Q}_n(k_1, \ldots, k_p) \leq e^{5\varepsilon(k_1 + \cdots + k_p)} \leq e^{\frac{5\varepsilon}{2}n}$, and hence the desired inequality follows. $\qquad\square$

Sublemmas 3.6 and 3.7 yield

$$|Q_n^p| = \sum_{K=1}^{[n/2]-1} \sum_{k_1+\cdots+k_p=K} |\mathcal{Q}_n(k_1, \ldots, k_p)| + \sum_{K=[n/2]}^{n} \sum_{k_1+\cdots+k_p=K} |\mathcal{Q}_n(k_1, \ldots, k_p)|$$

$$\leq \delta^{-1} e^{-\frac{\lambda}{5}n} \sum_{K=1}^{[n/2]-1} \#\left\{(k_1, \ldots, k_p): \sum_{j=1}^{p} k_j = K\right\} + \sum_{K=[n/2]}^{n} e^{-\frac{\lambda}{4}K} \#\left\{(k_1, \ldots, k_p): \sum_{j=1}^{p} k_j = K\right\}$$

$$\leq \delta^{-1} e^{-\frac{\lambda}{5}n} \sum_{K=1}^{[n/2]-1} e^{\beta(N)K} + \sum_{K=[n/2]}^{n} e^{-(\frac{\lambda}{4}-\beta(N))K} \leq \delta^{-1} e^{-\frac{\lambda}{9}n},$$

where $\beta(N) \to 0$ as $N \to \infty$. Hence

$$|\{R \geq n\}| \leq \sum_{p=1}^{n/N} |Q_n^p| \leq \frac{n}{N} \delta^{-1} e^{-\frac{\lambda}{9}n} \leq \delta^{-1} e^{-\frac{\lambda}{10}n}.$$

We also have

$$\#\{\omega \in \mathcal{Q}': R(\omega) = n\} \leq \sum_{p=1}^{n/N} \sum_{k_1+\cdots+k_p=n} \#\mathcal{Q}_n(k_1, \ldots, k_p) \leq \frac{n}{N} e^{5\varepsilon n} \leq e^{6\varepsilon n}.$$

The last inequality holds because $n \geq N$. This completes the proof of Lemma 3.3. $\qquad\square$

3.3. **Reduction to lower floors of the tower.** Using the return time function $R\colon \mathcal{Q}' \to \mathbb{N}$ we now introduce a Markov extension $\hat{f}\colon \Delta \circlearrowleft$ of $f\colon \bigcup_{i\geq 0} f^i\Omega_\infty \circlearrowleft$. Let

$$\Delta = \{(x, l): x \in \Omega_\infty, \quad l = 0, 1, \cdots, R(x) - 1\},$$

which we call a tower, and define

$$\hat{f}(x, l) = \begin{cases} (x, l+1) & \text{if } l+1 < R(x) \\ (f^{R(x)}x, 0) & \text{if } l+1 = R(x). \end{cases}$$

The point (x, l) is thought of as climbing the tower in the first case, and falling down from the tower in the second case. Define a projection $\pi\colon \Delta \to X$ by $\pi(x, l) = f^l x$. For $l \geq 0$, define $\Delta_l = \{(x, l) \in \Delta: x \in \{R > l\}\}$. Note that $\Delta_0 = \{(x, 0): x \in \Omega_\infty\}$. Let $\tau_l\colon \{R > l\} \to \Delta_l$ denote the canonical identification $\tau_l(x) = (x, l)$. By the results in Lemma 3.3 and the Folklore theorem [24], there exists an F-invariant probability measure ν_0 that is absolutely continuous

with respect to the Lebesgue measure on Ω_∞, with the density $\frac{d\nu_0}{d\mathrm{Leb}|\Omega_\infty}$ uniformly bounded from both sides and $\int R d\nu_0 < \infty$. Define a probability measure $\hat{\mu}$ on Δ by

$$\hat{\mu} = \frac{1}{\int R d\nu_0} \sum_{l=0}^\infty (\tau_l)_* \nu_0 |\{R > l\}.$$

Here, a measurable structure on Δ is chosen so that π is measurable. Observe that $\pi_* \hat{\mu} = \mu$.

We reduce the desired upper estimate in the proposition to an upper estimate on lower floors of the tower. For each $l \geq 0$, let $\mathcal{P}_l$ denote the restriction of the partition $\tilde{\mathcal{P}}_l$ to $\{R > l\}$. Using τ_l we transplant the partition $\mathcal{P}_l$ to Δ_l and also denote it by $\mathcal{P}_l$. Let $\mathcal{D}$ denote the resultant partition of Δ, namely $\mathcal{D} = \bigcup_{l \geq 0} \mathcal{P}_l$. Let $\hat{\varphi}_j = \varphi_j \circ \pi$. Let

$$\mathcal{B}_n = \left\{ A \in \bigvee_{i=0}^{n-1} \hat{f}^{-i} \mathcal{D} \colon \frac{1}{n} S_n \hat{\varphi}_j(x) \geq b_j \;\; j = 1, \ldots, d \text{ for some } x \in A \right\}.$$

Observe that

$$\frac{1}{n} \log \mu \left\{ \frac{1}{n} S_n \varphi_j \geq b_j \right\} = \frac{1}{n} \log \hat{\mu} \left\{ \frac{1}{n} S_n \hat{\varphi}_j \geq b_j \right\} \leq \frac{1}{n} \log |\mathcal{B}_n|,$$

where $|\mathcal{B}_n| = \sum_{A \in \mathcal{B}_n} \hat{\mu}(A)$. If $\mathcal{B}_n = \emptyset$ there is nothing to prove, and so we assume $\mathcal{B}_n \neq \emptyset$. Observe that

(20) $$2 \log 2 \geq \sup\{-F(\nu) \colon \nu \in \mathcal{M}_f\}.$$

Define

$$\mathcal{B}'_n = \bigcup_{l \leq \frac{10 \log 4}{\lambda} n} \{A \in \mathcal{B}_n \colon A \subset \Delta_l\} \;\; \text{and} \;\; \mathcal{B}''_n = \bigcup_{l > \frac{10 \log 4}{\lambda} n} \{A \in \mathcal{B}_n \colon A \subset \Delta_l\}.$$

Lemma 3.3(d) gives

(21) $$|\mathcal{B}''_n| \leq \sum_{l > \frac{10 \log 4}{\lambda} n} |\{R > l\}| \leq \delta^{-1} 4^{-n}.$$

Suppose $\mathcal{B}'_n = \emptyset$. If there is an ergodic measure $\nu \in \mathcal{M}_f$ with $\nu(\varphi_j) \geq b_j - \varepsilon$, $j = 1, \cdots, d$, then

$$\varlimsup_{n \to \infty} \frac{1}{n} \log |\mathcal{B}_n| = \varlimsup_{n \to \infty} \frac{1}{n} \log |\mathcal{B}''_n| \leq -2 \log 2 \leq F(\nu).$$

Hence, (1) holds. If there is no such ν, then (1) holds as well by the convention $\sup \emptyset = -\infty$ (see Sect.1). For the rest of the proof we assume $\mathcal{B}'_n \neq \emptyset$.

3.4. Approximation by points quickly falling down from the tower.

For $l \geq N$ and $\omega \in \mathcal{P}_l$, let $\tilde{\omega}$ denote the element of $\tilde{\mathcal{P}}_l$ containing ω, namely $\omega = \tilde{\omega} \cap \{R > l\}$. Points in ω may climb the tower for a very long period of time. The next lemma indicates that a positive definite fraction of points in $\tilde{\omega}$ quickly fall down from the tower. Let

$$C_0 = \frac{120}{\lambda}.$$

Lemma 3.8. *If* $l \geq \frac{\lambda N}{3\varepsilon}$ *and* $\omega \in \mathcal{P}_l$, *there exists* $\omega' \in \mathcal{Q}'$ *such that:*

(a) $\omega' \subset \tilde{\omega}$ *and* $|\omega'| \geq \frac{1}{3} e^{-10\delta^{-3} - 6C_0 \varepsilon^2 l} |\tilde{\omega}|$;

(b) $l \leq R(\omega') \leq (1 + \frac{4\varepsilon}{\lambda}) l$.

Proof. Let $l_0 = \min\{n \geq l \colon f^i\tilde{\omega} \text{ is free}\}$. If $f^l\tilde{\omega}$ is bound, let p denote the bound period. Since $d(0, f^n\tilde{\omega}) \geq \delta e^{-\varepsilon n}$ holds for $N \leq n \leq l$, Lemma 2.7 gives $p \leq \frac{3\varepsilon}{\lambda}l$, and thus $\max(p, N) \leq \frac{3\varepsilon}{\lambda}l$. Let $\mathcal{Q}$ denote the partition and S the stopping time obtained with Lemma 2.9 applied to $f^{l_0}\tilde{\omega}$. We have $|\{S \geq C_0\varepsilon l\}| \leq e^{-\frac{C_0\varepsilon\lambda}{9}l}|f^{l_0}\tilde{\omega}|$, or equivalently

$$(22) \qquad |\{S \leq C_0\varepsilon l\}| \geq (1 - e^{-\frac{C_0\varepsilon\lambda l}{9}})|f^{l_0}\tilde{\omega}|.$$

Sublemma 3.9. $|\omega| \geq \frac{1}{2}|\tilde{\omega}|$.

Proof. Choose a point $x \in \tilde{\omega} \cap \Omega_\infty$. By construction, $d(0, f^n x) \geq \delta e^{-\varepsilon n}$ holds for $N \leq n \leq l$. This means that the element of $\tilde{\mathcal{P}}_l$ containing x, which is $\tilde{\omega}$, is contained in Ω_l. An argument similar to that of the proof of Lemma 3.1 shows the desired inequality. $\qquad\square$

Sublemma 3.9 and the distortion estimate of Lemma 2.10 give

$$(23) \qquad |f^{l_0}\omega| \geq \frac{1}{2}e^{-10\delta^{-3}}|f^{l_0}\tilde{\omega}|.$$

By (22) (23) we have

$$|\{S \leq C_0\varepsilon l\} \cap f^{l_0}\omega| \geq \frac{1}{3}e^{-10\delta^{-3}}|f^{l_0}\tilde{\omega}|.$$

Since $\#\{\omega \in \mathcal{Q} \colon S(\omega) \leq C_0\varepsilon l\} \leq e^{6C_0\varepsilon^2 l}$ from Lemma 2.9, it is possible to choose an integer $r \leq C_0\varepsilon l$ and $\omega'' \in \mathcal{Q}$ such that $S(\omega'') = r$ and

$$|\omega'' \cap f^{l_0}\omega| \geq \frac{1}{3}e^{-10\delta^{-3}}e^{-6C_0\varepsilon^2 l}|f^{l_0}\tilde{\omega}|.$$

Let $\omega' = (f^r|\omega'' \circ f^{l_0}|\tilde{\omega})^{-1}\Omega_\infty^+$ or $= (f^r|\omega'' \circ f^{l_0}|\tilde{\omega})^{-1}\Omega_\infty^-$. Let $\widehat{\omega}'$ denote the smallest closed interval containing ω'. Since $\widehat{\omega}'$ is contained in an element of $\tilde{\mathcal{P}}_{l_0+r-1}$ and intersects Ω_∞, for all $N \leq n < l_0 + r$ we have $d(f^n\widehat{\omega}', 0) \geq \delta e^{-\varepsilon n}$. It follows that $\omega' \in \mathcal{Q}'$. We have $R(\omega') = l_0 + r \leq (1 + \frac{4\varepsilon}{\lambda})l$. $\qquad\square$

3.5. **Construction of a horseshoe.** Let $L_1, \ldots, L_q$ be a collection of pairwise disjoint closed intervals in (c_1, c_0) and $\xi > 0$ be an integer such that:

(H1) for each $1 \leq i \leq q$, $f^\xi|\text{int}L_i$ is a diffeomorphism and $f^\xi L_i = [c_1, c_0]$;

(H2) there exists $C > 0$ such that for all $1 \leq i \leq q$ and $x, y \in L_i$, $\log\frac{J^\xi(x)}{J^\xi(y)} \leq C|f^\xi x - f^\xi y|^{\varepsilon^2}$;

(H3) there exists $\kappa > 1$ such that for all $x \in \bigcup_{i=1}^q L_i$, $J^\xi(x) \geq \kappa$.

We say $L_1, \ldots, L_q$ *generates a horseshoe for* f^ξ. Let

$$H(\xi; L_1, \ldots, L_q) = \bigcap_{i \geq 0}(f^\xi)^{-i}\left(\bigcup_{i=1}^q L_i\right).$$

Observe that $f^\xi \colon H(\xi; L_1, \ldots, L_q) \circlearrowleft$ is Hölder conjugate to the one-sided shift on q-symbols.

We construct a horseshoe which ties in with $|\mathcal{B}'_n|$. To interpolate between $\hat{\mu}$ and the Lebesgue on Ω_∞ we use the uniform boundedness of the density of ν_0. Let $\hat{C}_1 > 0$, $\hat{C}_2 > 0$ be such that $\hat{C}_1 \leq \frac{d\nu_0}{d\text{Leb}|\Omega_\infty} \leq \hat{C}_2$. Set $C_i = \hat{C}_i \int R d\nu_0$, $i = 1, 2$. Define $\iota \colon \Delta \to X$ by $\iota(x, l) = x$. For any measurable set $B \subset \Delta_l$ we have $\hat{\mu}(B) = \int R\nu_0 \cdot \nu_0(B)$, and thus

$$(24) \qquad C_1|\iota(B)| \leq \hat{\mu}(B) \leq C_2|\iota(B)|.$$

Set

$$C_3 = \frac{36 \log 4}{\lambda^2}, \quad C_4 = \frac{40 \log 4}{\lambda^2}, \quad C_5 = \frac{\lambda^2 C_1 C_2^{-2}}{240 \log 4}.$$

Lemma 3.10. *For all large n there exist a collection $L_1, \ldots, L_q$ of closed intervals in (c_1, c_0) and an integer $\xi \in [(1 - C_3 \varepsilon)n, (1 + C_4 \varepsilon)n]$ such that:*

(a) *$L_1, \ldots, L_q$ generate a horseshoe for f^ξ;*

(b) *$\sum_{i=1}^q |L_i| \geq \frac{C_5 \delta e^{-10\delta^{-3}}}{\varepsilon n} e^{-6C_0 \varepsilon^2 \left(\frac{10 \log 4}{\lambda} + 1 \right)n} |\mathcal{B}_n'|$;*

(c) *for all $x \in H$, $\frac{1}{\xi} S_\xi \varphi_j(x) \geq b_j - \sqrt{\varepsilon}$, $j = 1, \ldots, d$.*

Proof. Let $A \in \mathcal{B}_n'$ and suppose $A \subset \Delta_{l_A}$. By definition, $\hat{f}^n A$ is an element of $\mathcal{D}|\Delta_{l_A+n}$, and thus $\iota(\hat{f}^n A)$ is an element of $\mathcal{P}_{l_A+n}$, which we denote by ω_A. Take $\omega_A' \subset \tilde{\omega}_A \in \tilde{\mathcal{P}}_{l_A+n}$ for which the conclusions of Lemma 3.8 holds. Set

$$t_A = R(\omega_A') - \min\{k \geq l_A : f^k \tilde{\omega}_A \text{ is free}\}.$$

We estimate t_A. Lemma 3.8(b) gives

$$l_A + n \leq R(\omega_A') \leq \left(1 + \frac{4\varepsilon}{\lambda} \right)(l_A + n),$$

and we have

$$l_A \leq \min\{k \geq l_A : f^k \tilde{\omega}_A \text{ is free}\} \leq \left(1 + \frac{3\varepsilon}{\lambda} \right) l_A.$$

These two estimates and $l_A \leq \frac{10 \log 4}{\lambda} n$ which follows from $A \in \mathcal{B}_n'$ yield

$$t_A \leq \left(1 + \frac{4\varepsilon}{\lambda} \right)(l_A + n) - l_A \leq \left(1 + \frac{4\varepsilon}{\lambda} + \frac{40 \log 4 \cdot \varepsilon}{\lambda^2} \right) n,$$

and

$$t_A \geq l_A + n - \left(1 + \frac{3\varepsilon}{\lambda} \right) l_A \geq \left(1 - \frac{30 \log 4 \cdot \varepsilon}{\lambda^2} \right) n.$$

Let $A' \subset A$ be such that $\iota(\hat{f}^n A') = \omega_A'$. The above estimates of t_A allo us to choose an integer $\xi_0 \in [(1 - \frac{35 \log 4 \cdot \varepsilon}{\lambda^2})n, (1 + \frac{45 \log 4 \cdot \varepsilon}{\lambda^2})n]$ such that

$$(25) \qquad \sum_{A \in \mathcal{B}_n' : t_A = \xi_0} \hat{\mu}(A') \geq \frac{\lambda^2}{80 \log 4 \cdot \varepsilon n} \sum_{A \in \mathcal{B}_n'} \hat{\mu}(A').$$

Set $q = \sharp\{A \in \mathcal{B}_n' : t_A = \xi_0\}$ and write $\{A \in \mathcal{B}_n' : t_A = \xi_0\} = \{A_1, \ldots, A_q\}$. For $1 \leq i \leq q$, let $l_i = \min\{n \geq l_{A_i} : f^k \tilde{\omega}_{A_i} \text{ is free}\}$. By (A4), it is possible to choose an integer $\eta > 0$ and a closed interval $I^+ \subset \Lambda^+$ such that f^η sends the interiors of I^+ diffeomorphically onto (c_1, c_0). Let $I^- = -I^+$. We now define $L_i := f^{l_i} \widehat{\iota(A_i')}$, where $\widehat{\iota(A_i')}$ is the smallest closed interval containing $\iota(A_i')$. Then L_i is a closed interval in (c_1, c_0), and $f^{\xi_0} L_i = \Lambda^+$ or $= \Lambda^-$. By construction, $L_1, \ldots, L_q$ are pairwise disjoint. Let $\xi = \xi_0 + \eta$. Then $\xi \in [(1 - C_3 \varepsilon)n, (1 + C_4 \varepsilon)n]$ holds for sufficiently large n. For any $x \in \bigcup_{i=1}^q L_i$ we have $J^\xi(x) = J^\eta(f^{\xi_0} x) J^{\xi_0}(x) \geq \min_{y \in \Lambda} J^\eta(y) \cdot e^{\frac{\lambda}{3} \xi_0} > 1$. Hence, $L_1, \ldots, L_q$ generate a horseshoe for f^ξ.

For any pair (A, A') as above we have

$$\hat{\mu}(A') = \hat{\mu}(\hat{f}^n A') \geq C_1|\omega'_A| \geq \frac{C_1}{3}e^{-10\delta^{-3}-6C_0\varepsilon^2(l_A+n)}|\tilde{\omega}_A| \geq \frac{C_1}{3}e^{-10\delta^{-3}-6C_0\varepsilon^2(l_A+n)}|\omega_A|$$

$$\geq \frac{C_1 C_2^{-1}}{3}e^{-10\delta^{-3}-6C_0\varepsilon^2(\frac{10\log 4}{\lambda}+1)n}\hat{\mu}(\hat{f}^n A) = \frac{C_1 C_2^{-1}}{3}e^{-10\delta^{-3}-6C_0\varepsilon^2(\frac{10\log 4}{\lambda}+1)n}\hat{\mu}(A),$$

where we have used: (24) for the first inequality; Lemma 3.8 for the second; $l_A \leq \frac{10}{\lambda}n$ and $\omega_A \subset \tilde{\omega}_A$ for the third; (24) again for the fourth. Plugging this estimate into the right-hand-side of (25) we have

$$(26) \qquad \sum_{i=1}^q \hat{\mu}(A'_i) \geq \frac{\lambda^2 C_1 C_2^{-1} e^{-10\delta^{-3}-6C_0\varepsilon^2(\frac{10\log 4}{\lambda}+1)n}}{240\log 4 \cdot \varepsilon n}|\mathcal{B}'_n|.$$

Meanwhile we have,

$$(27) \qquad |L_i| = |f^{l_i}\widehat{\iota(A'_i)}| \geq \delta|\widehat{\iota(A'_i)}| \geq \delta|\iota(A'_i)| \geq C_2^{-1}\delta\hat{\mu}(A'_i).$$

Summing (27) over all $1 \leq i \leq q$ and combining the result with (26) gives (b).

To prove (d) we need the next

Sublemma 3.11. *For any $n \geq 1$ and $\omega \in \tilde{\mathcal{P}}_{n-1}$, $\sum_{i=0}^{n-1}|f^i\omega| \leq 10\delta^{-1}$.*

We finish the proof of (d) assuming the conclusion of this sublemma. It suffices to show $S_\xi\hat{\varphi}_j(x) \geq (b_j - \sqrt{\varepsilon})\xi$ for all $x \in \bigcup_{i=1}^q(\hat{f})^{l_i-l_{A_i}}A_i$. In view of the definition of the partition $\mathcal{B}_n$, pick $x_i \in A_i$ such that $S_n\hat{\varphi}(x_i) \geq b_j n$ holds for $j = 1,\ldots,d$. We have

$$|S_\xi\hat{\varphi}_j(\hat{f}^{l_i-l_{A_i}}x_i) - S_n\hat{\varphi}_j(x_i)| \leq (2(l_i - l_{A_i}) + |\xi - n|)\|\varphi_j\|,$$

where $\|\varphi_j\| = \sup|\varphi_j|$. Hence

$$S_\xi\hat{\varphi}_j(\hat{f}^{l_i-l_{A_i}}x_i) \geq S_n\hat{\varphi}_j(x_i) - (2(l_i - l_{A_i}) + |\xi - n|)\|\varphi_j\| \geq b_j n - \left(\frac{60\log 4}{\lambda^2} + C_4\right)\varepsilon n,$$

where we have used $|l_i - l_{A_i}| \leq \frac{3\varepsilon}{\lambda}l_{A_i} \leq \frac{30\log 4 \cdot \varepsilon}{\lambda^2}n$ and $|\xi - n| \leq C_4\varepsilon n$ for the last inequality. Hence, for each $i = 1,\ldots,q$ and $j = 1,\ldots,d$,

$$(28) \qquad S_\xi\hat{\varphi}_j(\hat{f}^{l_i-l_{A_i}}x_i) \geq \left(b_j - \frac{\sqrt{\varepsilon}}{2}\right)\xi.$$

By Sublemma 3.11, for any $x \in \hat{f}^{l_i-l_{A_i}}A_i$ we have $|S_n\hat{\varphi}_j(f^{l_i-l_{A_i}}x_i) - S_n\hat{\varphi}_j(x)| \leq \text{Lip}(\varphi_j)\cdot 10\delta^{-1}$, where $\text{Lip}(\varphi_j)$ denotes the Lipschitz constant of φ_j. Hence we have

$$(29) \qquad |S_\xi\hat{\varphi}_j(\hat{f}^{l_i-l_{A_i}}x_i) - S_\xi\hat{\varphi}_j(x)| \leq \text{Lip}(\varphi_j)\cdot 10\delta^{-1} + 2\|\varphi_j\|(\xi - n) \leq \frac{\sqrt{\varepsilon}}{2}\cdot\xi.$$

(28) (29) give $S_\xi\hat{\varphi}_j(x) \geq (b_j - \sqrt{\varepsilon})\xi$.

It is left to prove Sublemma 3.11. Let $n_1 < \cdots < n_s < n$ denote all the free returns in the first $n-1$-iterates of ω, with $p_1, \cdots, p_s$ the corresponding bound periods. Let $1 \leq i \leq s$. For each $j \in [n_i + 1, n_i + p_i - 1]$, choose $\theta_j \in f^{n_i}\omega$ such that $|f^j\omega| = |f^{n_i}\omega|\cdot J^{j-n_i}(\theta_j)$. We have $|f^{j-n_i}\theta_j - f^{j-n_i}0| \leq e^{-\varepsilon(p_i-1)}$. By the bounded distortion during the bound period,

$$J^{j-n_i-1}(f\theta_j) \leq 2\cdot\frac{|f^{j-n_i}\theta_j - f^{j-n_i}0|}{|f\theta_j - f0|} \leq \frac{e^{-\varepsilon(p_i-1)}}{\gamma_{p_i}^2} \leq \frac{3e^{-\varepsilon(p_i-1)}e^{\alpha\sqrt{p_i}}}{\gamma_{p_i-1}^2}.$$

For the last inequality we have used (9). We also have $|f'\theta_j| \leq 4\gamma_{p_i-1}$. Plugging these two derivative estimates into the equality and summing the result over all j gives

$$\sum_{j=n_i+1}^{n_i+p_i-1} |f^j\omega| \leq |f^{n_i}\omega| \cdot \frac{e^{-\frac{\varepsilon}{2}(p_i-1)}}{\gamma_{p_i-1}}.$$

Summing this over all i gives

$$\sum_{i=1}^{s}\sum_{j=n_i+1}^{n_i+p_i-1} |f^j\omega| \leq \sum_{i=1}^{s}|f^{n_i}\omega| \cdot \frac{e^{-\frac{\varepsilon}{2}(p_i-1)}}{\gamma_{p_i-1}} = \sum_{p\geq N}\frac{e^{-\frac{\varepsilon}{2}(p-1)}}{\gamma_{p-1}}\sum_{i:\,p_i=p}|f^{n_i}\omega|.$$

We prove for any p,

(30)
$$\sum_{i:\,p_i=p}|f^{n_i}\omega| \leq \gamma_{p-1}.$$

Substituting this estimate into the previous inequality gives

$$\sum_{i=1}^{s}\sum_{j=n_i+1}^{n_i+p_i-1} |f^j\omega| \leq \sum_{p\geq N} e^{-\frac{\varepsilon}{2}(p-1)} \leq 1.$$

Meanwhile, for contributions from free segments, in view of (13) we have

$$\sum_{i=0}^{s-1}\sum_{j=n_i+p_i}^{n_{i+1}} |f^j\omega| \leq 10\delta^{-3}.$$

These two inequalities yield the desired one.

We prove (30). Let n_{i_j}, $j = 1, \ldots, m$ denote the subsequence of returns with the same bound period equal to p. By construction $|f^{n_{i_m}}\omega| \leq 2\gamma_{p-1}$ holds. Using Lemma 2.3 and Lemma 2.5, for all $\theta \in f^{n_{i_j}}\omega$ we have $J^{n_{i_m}-n_{i_j}}(\theta) \geq e^{\frac{\lambda p}{3}(m-j)}$. Hence $|f^{n_{i_j}}\omega| \leq e^{-\frac{\lambda p}{3}(m-j)}|f^{n_{i_m}}\omega|$, and

$$\sum_{j=1}^{m}|f^{n_{i_j}}\omega| \leq \sum_{j=1}^{m} e^{-\frac{\lambda p}{3}(m-j)}|f^{n_{i_m}}\omega| \leq 2|f^{n_{i_m}}\omega|.$$

This completes the proof of Lemma 3.11 and hence that of Lemma 3.10. □

3.6. **Construction of a measure on the horseshoe.** We are in position to construct a measure for which (1) (2) hold. Define a potential $\Phi\colon H \to \mathbb{R}$ by $\Phi(x) = \log J^\xi(x)$. By (H2), Φ is Hölder continuous. Let $g = f^\xi|H$. Let τ denote the equilibrium state of g for Φ, namely the unique g-invariant measure such that $h(\tau; g) - \tau(\Phi) = \sup\{h(\nu; g) - \nu(\Phi)\colon \nu \text{ is } g\text{-invariant}\}$. Here, $h(\nu; g)$ denotes the entropy of ν. Let $\sigma = \frac{1}{\xi}\sum_{l=0}^{\xi-1}(f^l)_*\tau$, which is f-invariant and ergodic. From Lemma 3.10(c) it follows that $S_\xi\varphi_j \geq (b_j - \sqrt{\varepsilon})\xi$ τ-a.e. Hence $\sigma(\varphi_j) = \frac{1}{\xi}\tau(S_\xi\varphi_j) \geq b_j - \sqrt{\varepsilon}$, and (2) holds.

We prove (2). Set $C_6 = e^{C\sum_{n=0}^{\infty}\kappa^{-n}}$, where $C > 0$, $\kappa > 1$ are the constants in (H2) (H3). For $k > 0$ and $\{a_0, \ldots, a_k\} \subset \{1, \ldots, q\}$, let

$$[a_0 \cdots a_k] = L_{a_0} \cap g^{-1}L_{a_1} \cap \cdots \cap g^{-k}L_{a_k}.$$

We have $\frac{|[a_0 \cdots a_k]|}{|[a_0 \cdots a_{k-1}]|} \geq C_6 |L_{a_k}|$, and thus

$$\sum_{\{a_0, \cdots, a_k\}} |[a_0 \cdots a_k]| = \sum_{\{a_0, \cdots, a_{k-1}\}} |[a_0 \cdots a_{k-1}]| \sum_{a_k} \frac{|[a_0 \cdots a_k]|}{|[a_0 \cdots a_{k-1}]|}$$

$$\geq C_6 \sum_{j=1}^{q} |L_j| \sum_{\{a_0, \cdots, a_{k-1}\}} |[a_0 \cdots a_{k-1}]| \geq \left(C_6 \sum_{j=1}^{q} |L_j| \right)^{k+1}.$$

This yields

$$(31) \qquad \varliminf_{k \to \infty} \frac{1}{k} \log \sum_{\{a_0, \cdots, a_k\}} |[a_0 \cdots a_k]| \geq \log \sum_{j=1}^{q} |L_j| + \log C_6.$$

Let $\nu_{a_0 \cdots a_k}$ denote the atomic probability measure uniformly distributed on the periodic orbit of period $k+1$ in $[a_0 \cdots a_k]$. The bounded distortion gives

$$(32) \qquad |[a_0 \cdots a_k]| \leq C_6 \cdot e^{-(k+1)\nu_{a_0 \cdots a_k}(\Phi)}.$$

Define a probability measure ν_k on H by

$$\nu_k = \rho_k \sum_{\{a_0, \cdots, a_k\}} |[a_0 \cdots a_k]| \nu_{a_0 \cdots a_k},$$

where ρ_k is the normalizing constant. Pick an accumulation point of the sequence (ν_k) and denote it by ν_∞. Passing to proper subsequences if necessary we may assume this convergence takes place for the entire sequence. Using $\nu_k[a_0 \cdots a_k] = \rho_k |[a_0 \cdots a_k]|$ and (32) we have

$$\log \sum_{\{a_0, \cdots, a_k\}} |[a_0 \cdots a_k]| = \sum_{\{a_0, \cdots, a_k\}} \nu_k[a_0 \cdots a_k] \left(-\log \nu_k[a_0 \cdots a_k] + \log |[a_0 \cdots a_k]| \right)$$

$$\leq - \sum_{\{a_0, \cdots, a_k\}} \nu_k[a_0 \cdots a_k] \log \nu_k[a_0 \cdots a_k] - (k+1)\nu_k(\Phi) + \log C_6.$$

The usual proof of the variational principle [[32] Theorem 9.10.] shows

$$(33) \qquad \varlimsup_{k \to \infty} \frac{1}{k} \log \sum_{\{a_0, \cdots a_k\}} |[a_0 \cdots a_k]| \leq h(\nu_\infty; g) - \nu_\infty(\Phi).$$

Combining (31) (33) and then using (26), for all large n we have

$$h(\nu_\infty; g) - \nu_\infty(\Phi) \geq \log \sum_{i=1}^{q} |L_i| \geq -130 C_0 \varepsilon^2 n - \log(\varepsilon n) + \log |\mathcal{B}'_n|.$$

Since $F(\sigma) \leq 0$ and $\xi \in [(1 - C_3\varepsilon)n, (1 + C_4\varepsilon)n]$ we have

$$n \cdot F(\sigma) \geq \frac{\xi}{1 - C_3\varepsilon} \cdot F(\sigma) = \frac{1}{1 - C_3\varepsilon} \left(h(\tau; g) - \tau(\Phi) \right)$$

$$\geq \frac{1}{1 - C_3\varepsilon} \left(h(\nu_\infty; g) - \nu_\infty(\Phi) \right) \geq \frac{1}{1 - C_3\varepsilon} \left(-130 C_0 \varepsilon^2 n - \log(\varepsilon n) + \log |\mathcal{B}'_n| \right).$$

Rearranging this and then combining the result with $\frac{1}{n}\log|\mathcal{B}_n''| \leq F(\sigma)$ which follows from (21), for sufficiently large n we obtain

$$\frac{1}{n}\log|\mathcal{B}_n| \leq (1 - C_3\varepsilon)F(\sigma) + 130C_0\varepsilon^2 + \frac{1}{n}\log(\varepsilon n).$$

Since $C_3 < 200$, (1) holds. This completes the proof of the proposition.

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